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# Generalized Campanato spaces with variable growth condition

By

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**Abstract**

This is a survey on generalized Campanato spaces with variable growth condition. We first define generalized Campanato spaces and related function spaces. Then we state the relations among these function spaces and the characterization of pointwise multipliers on generalized Campanato spaces. Next we state the boundedness of singular integral operators and the convolution operator with the heat kernel. We also give an application of generalized Campanato spaces to the Cauchy problem for the Navier-Stokes equation. Finally, we state the boundedness of the commutators generated by functions in generalized Campanato spaces.

## § 1. Introduction

This is a survey on generalized Campanato spaces with variable growth condition.

The Campanato space was introduced and studied by Campanato [4, 5] (1963, 1964), Peetre [39, 40] (1966, 1969), Stampacchia [45] (1965), Spanne [44] (1965), etc. Since then, the theory of Campanato spaces plays an important role in harmonic analysis and partial differential equations.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius $r$, that is, $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and $\chi_G$ the Lebesgue measure of $G$ and the characteristic function of $G$, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B$, let

$$f_B = ⨍_B f = ⨍_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

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Then the Campanato space $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ is defined as follows:

**Definition 1.1.** For $p \in [1,\infty)$ and $\lambda \in [-n/p, 1]$, let $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ be the set of all functions $f$ such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\lambda}(\mathbb{R}^n)} = \sup_{B=B(x,r)} \frac{1}{r^\lambda} \left( \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

We regard $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ as a space of functions modulo null-functions and constant functions. Then $\|f\|_{\mathcal{L}_{p,\lambda}(\mathbb{R}^n)}$ is a norm and thereby $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\lambda \in (0,1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\lambda(\mathbb{R}^n)$ modulo null-functions. By the John-Nirenberg inequality we conclude that, if $\lambda \in [0,1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathcal{L}_{1,\lambda}(\mathbb{R}^n)$ with equivalent norms for each $p$. If $\lambda \in [-n/p, 0)$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ coincides with the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ modulo constant functions, which is defined by the norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{B=B(x,r)} \frac{1}{r^\lambda} \left( \fint_B |f(y)|^p \, dy \right)^{1/p}.$$

If $\lambda = -n/p$, then $L_{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For these relations, see [4, 17, 24, 40].

The generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ with variable growth condition is defined as follows: For a variable growth function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$ we write $\phi(B) = \phi(x,r)$.

**Definition 1.2.** For $p \in [1,\infty)$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ be the set of all functions $f$ such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)} = \sup_{B} \frac{1}{\phi(B)} \left( \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

We regard $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ as a space of functions modulo null-functions and constant functions. Then $\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)}$ is a norm and thereby $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is a Banach space. If $\phi(x,r) = r^\lambda$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the usual Campanato space $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. For example, let $\lambda(\cdot) : \mathbb{R}^n \to [-n/p, 1]$ be a continuous function and

$$\phi(x,r) = r^{\lambda(x)}, \quad \lambda(x) = \begin{cases} 0 & \text{on } B_1, \\ 1 & \text{on } B_2, \\ -n/p & \text{on } B_3. \end{cases}$$

In this case, if $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, then $f$ is a BMO function on $B_1$, a Lipschitz function on $B_2$ and an $L^p$ function on $B_3$.

The generalized Campanato space $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ was introduced by the author and Yabuta [36] (1985) to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ and studied in [22, 23, 25, 27, 28, 31, 36, 48], etc.

Let $L^0(\mathbb{R}^n)$ be the set of all measurable functions on $\mathbb{R}^n$. Let $E_1, E_2 \subset L^0(\mathbb{R}^n)$ be subspaces. We say that a function $g \in L^0(\mathbb{R}^n)$ is a pointwise multiplier from $E_1$ to $E_2$, if the pointwise multiplication $fg$ is in $E_2$ for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from $E_1$ to $E_2$. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$.

To consider the pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ we introduce a norm

$$\|f\|_{\mathrm{BMO}^\natural(\mathbb{R}^n)} = \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + |f_{B(0,1)}|.$$

Then $\mathrm{BMO}^\natural(\mathbb{R}^n) = (\mathrm{BMO}(\mathbb{R}^n), \|\cdot\|_{\mathrm{BMO}^\natural(\mathbb{R}^n)})$ is a Banach space not modulo constant functions. We have the following theorem:

**Theorem 1.3** ([36] (1985)). *Let*

$$\phi(x,r) = \frac{1}{\log(r + 1/r + |x|)}, \quad x \in \mathbb{R}^n,\ r \in (0,\infty).$$

*Then* $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n)) = \mathcal{L}_{1,\phi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *In this case, the operator norm of* $g \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ *is comparable to* $\|g\|_{\mathcal{L}_{1,\phi}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}$.

For example,

$$g_1(x) = \sin\left(\chi_{B(0,1/e)}(x) \log\log(|x|^{-1})\right) \tag{1.1}$$

and

$$g_2(x) = \sin\left(\chi_{B(0,e)^\complement}(x) \log\log|x|\right) \tag{1.2}$$

are in $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$. Note that $g_1$ is not continuous and that $\lim\limits_{|x|\to\infty} g_2(x)$ does not exist. The example (1.1) was given by Janson [15] (1976) and Stegenga [46] (1976) on the Torus $\mathbb{T}^n$, and the example (1.2) by [36] on $\mathbb{R}^n$.

Twenty years later, Theorem 1.3 was used by Lerner [21] (2005) to study the class $\mathcal{P}(\mathbb{R}^n)$ of functions for which the Hardy-Littlewood maximal operator is bounded on the Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, and positively solve a conjecture by Deining [11] saying that there are discontinuous functions belonging to $\mathcal{P}(\mathbb{R}^n)$. He proved the following theorem:

**Theorem 1.4** ([21] (2005)). *Let $p(\cdot)$ be a real valued measurable function. If $p(\cdot) \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, then there exists a positive constant $\alpha$ such that the Hardy-Littlewood maximal operator $M$ is bounded on $L^{\alpha+p(\cdot)}(\mathbb{R}^n)$.*

Later, the author and Sawano [34] (2012) proved that $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. In general the predual is not unique. See [30, 33] (2008, 2017) for another predual of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$.

This paper is organized as follows. In Section 2, we state the relations between $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and Morrey or Hölder spaces with variable growth condition. In Sections 3 and 4, we state the characterization of pointwise multipliers on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and their examples, respectively. In Sections 5 and 6 we consider the boundedness of singular integral operators and the convolution operator with the heat kernel on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, respectively. In Section 7, we give an application of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ to the Cauchy problem for the Navier-Stokes equation. In Section 8, we state on generalized fractional integral operators on generalized Morrey spaces with variable growth condition. Finally, in Section 9, we state the boundedness of the commutators $[b, T]$ and $[b, I_\rho]$ on generalized Morrey spaces with variable growth condition, where $T$ is a singular integral operator, $I_\rho$ is a generalized fractional integral operator and $b \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$.

## § 2. Related function spaces

In this section, we state the relations between $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and other function spaces with variable growth condition.

**Definition 2.1.** For $1 \le p < \infty$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the function spaces $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ are the sets of all functions $f$ such that

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B(0,1)}| < \infty,$$
$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \fint_B |f(x)|^p \, dx \right)^{1/p} < \infty,$$

respectively.

We regard $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions. Then these functionals are also norms and thereby these spaces are Banach spaces. If $\phi(B) = |B|^{-1/p}$ for all balls $B$, then

$$\|f\|_{L_{p,\phi}} = \|f\|_{L^p}.$$

From the definition it follows that

$$\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{L_{p,\phi}}, \quad \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le (2 + \phi(0,1))\|f\|_{L_{p,\phi}}.$$

**Definition 2.2.** For $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the function spaces $\Lambda_\phi(\mathbb{R}^n)$ and

$\Lambda^{\natural}_{\phi}(\mathbb{R}^n)$ are the sets of all functions $f$ such that

$$\|f\|_{\Lambda_\phi} = \sup_{x,y\in\mathbb{R}^n,\ x\neq y} \frac{2|f(x)-f(y)|}{\phi(x,|x-y|)+\phi(y,|y-x|)} < \infty,$$
$$\|f\|_{\Lambda^{\natural}_{\phi}} = \|f\|_{\Lambda_\phi} + |f(0)| < \infty,$$

respectively.

We regard $\Lambda^{\natural}_{\phi}(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$, and $\Lambda_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$ modulo constant functions. Then these functionals are also norms and thereby these spaces are Banach spaces. For $\phi(x,r) = r^\alpha$, $0 < \alpha \le 1$, we denote $\Lambda_{r^\alpha}(\mathbb{R}^n)$ and $\Lambda^{\natural}_{r^\alpha}(\mathbb{R}^n)$ by $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ and $\mathrm{Lip}^{\natural}_\alpha(\mathbb{R}^n)$, respectively. In this case,

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup_{x,y\in\mathbb{R}^n,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \quad \text{and} \quad \|f\|_{\mathrm{Lip}^{\natural}_\alpha} = \|f\|_{\mathrm{Lip}_\alpha} + |f(0)|.$$

If $\phi(x,r) = \min(r^\alpha, 1)$, $0 < \alpha \le 1$, then

$$\|f\|_{\Lambda^{\natural}_{\phi}} \sim \|f\|_{\mathrm{Lip}_\alpha} + \|f\|_{L^\infty}.$$

For two variable growth functions $\phi_1$ and $\phi_2$, we write $\phi_1 \sim \phi_2$ if there exists a positive constant $C$ such that

$$C^{-1}\phi_1(B) \le \phi_2(B) \le C\phi_1(B) \quad \text{for all balls } B.$$

In this case, two function spaces defined by $\phi_1$ and by $\phi_2$ coincide with equivalent norms.

We consider the following conditions on variable growth function $\phi$:

$$\frac{1}{A_1} \le \frac{\phi(x,s)}{\phi(x,r)} \le A_1, \qquad \text{if } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.1}$$

$$\frac{1}{A_2} \le \frac{\phi(x,r)}{\phi(y,r)} \le A_2, \qquad \text{if } |x-y| \le r, \tag{2.2}$$

$$\phi(x,r) \le A_3\phi(x,s), \qquad \text{if } r < s, \tag{2.3}$$

$$A_4\phi(x,r) \ge \phi(x,s), \qquad \text{if } r < s, \tag{2.4}$$

where $A_i$ $(i = 1,2,3,4)$ are positive constants independent of $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$. The conditions (2.1), (2.3) and (2.4) are called the doubling, almost increasingness and almost decreasingness conditions, respectively. The condition (2.2) is introduced in [25] and studied in [34] precisely. In this paper, we call it the nearness or compatible condition.

Note that (2.2) and (2.3) imply that there exists a positive constant $C$ such that, for all $x, y \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\phi(x,r) \le C\phi(y,s) \quad \text{if} \quad B(x,r) \subset B(y,s).$$

Then we have the following three theorems:

**Theorem 2.3** ([30] (2008)). *If $\phi$ satisfies* (2.1), (2.2) *and* (2.3)*, then, for every $1 \le p < \infty$, $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{\natural}_{1,\phi}(\mathbb{R}^n)$ with equivalent norms, respectively.*

**Theorem 2.4** ([28] (2006)). *If $\phi$ satisfies* (2.1), (2.2) *and* (2.3)*, and if there exists a positive constant $C$ such that*

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r \in (0,\infty), \tag{2.5}$$

*then, for every $1 \le p < \infty$, each element in $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ can be regarded as a continuous function, (that is, each element is equivalent to a continuous function modulo null-functions) and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_{\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \Lambda^{\natural}_{\phi}(\mathbb{R}^n)$ with equivalent norms, respectively. In particular, if $\phi(x,r) = r^{\alpha}$, $0 < \alpha \le 1$, then, for every $1 \le p < \infty$, $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}^{\natural}_{\alpha}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_{\alpha}(\mathbb{R}^n)$ with equivalent norms, respectively.*

**Theorem 2.5** ([28] (2006)). *Let $1 \le p < \infty$. If $\phi$ satisfies* (2.1) *and* (2.2)*, and if there exists a positive constant $C$ such that*

$$\int_r^{\infty} \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r \in (0,\infty), \tag{2.6}$$

*then, for $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, the limit $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$ exists and*

$$\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - \sigma(f)\|_{L_{p,\phi}}.$$

*That is, the mapping $f \mapsto f - \sigma(f)$ is bijective and bicontinuous from $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$* (*modulo constants*) *to $L_{p,\phi}(\mathbb{R}^n)$.*

These theorems are valid for spaces of homogeneous type, see [28, 30].

## § 3. Pointwise multipliers on Campanato spaces

Theorem 1.3 was extended to $\mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))$ by [25], to spaces of homogeneous type by [27, 36], to weighted BMO by [48], to Campanato spaces on the Gauss measure space by [23], to RBMO on non-doubling measure spaces by [22], etc.

In this section, we investigate the pointwise multipliers on generalized Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$. We denote $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{1,\phi}(\mathbb{R}^n)$ by $\mathrm{BMO}_{\phi}(\mathbb{R}^n)$ and $\mathrm{BMO}^{\natural}_{\phi}(\mathbb{R}^n)$, respectively, for $p = 1$ and variable growth function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$.

**Theorem 3.1** ([25] (1993)). *Let $1 \le p < \infty$. Assume that $\phi$ satisfies* (2.1) *and* (2.2)*, that $r \mapsto \phi(x,r)/r$ is almost decreasing and that there exists a positive constant $A$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\int_0^r \frac{\phi(x,t)t^{n/p}}{t}\,dt \le A\phi(x,r)r^{n/p}. \tag{3.1}$$

*Let*

$$\Phi^*(x,r) = \int_1^{\max(2,|x|,r)} \frac{\phi(0,t)}{t}\,dt,\ \ \Phi^{**}(x,r) = \int_r^{\max(2,|x|,r)} \frac{\phi(x,t)}{t}\,dt, \tag{3.2}$$

*and let $\psi = \phi/(\Phi^* + \Phi^{**})$. Then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)) = \mathcal{L}_{p,\psi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

*and*

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)},$$

*where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))$.*

*Remark* ([25] (1993)). Under the assumption in Theorem 3.1, let

$$f_a(x) = \int_{|x-a|}^1 \frac{\phi(a,t)}{t}\,dt.$$

Then $f_a$ is in $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ for all $a \in \mathbb{R}^n$.

For $\phi_i$ $(i = 1, 2)$, we define $\Phi_i^*$ and $\Phi_i^{**}$ by (3.2).

**Theorem 3.2** ([27] (1997)). *Let $1 < p < \infty$. Assume that $\phi_i$ $(i = 1, 2)$ satisfy* (2.1), (2.2), (2.3) *and* (3.1) *and that $r \mapsto \phi_i(x,r)/r$ $(i = 1, 2)$ are almost decreasing. Assume also that there exists a positive constant $A$ such that, for all $r \in (1,\infty)$,*

$$\int_1^r \frac{\phi_2(x,t)}{\phi_1(x,t)} t^{n/p-1}\,dt \le A\frac{\phi_2(x,r)}{\phi_1(x,r)} r^{n/p}. \tag{3.3}$$

*Let $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$. Then*

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{\phi_1}(\mathbb{R}^n), \mathrm{BMO}^{\natural}_{\phi_2}(\mathbb{R}^n)) = \mathrm{BMO}_{\phi_3}(\mathbb{R}^n) \cap L_{1,\phi_2/\phi_1}(\mathbb{R}^n)$$

*and*

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathrm{BMO}_{\phi_3}(\mathbb{R}^n)} + \|g\|_{L_{1,\phi_2/\phi_1}(\mathbb{R}^n)},$$

*where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathrm{BMO}^{\natural}_{\phi_1}(\mathbb{R}^n), \mathrm{BMO}^{\natural}_{\phi_2}(\mathbb{R}^n))$.*

In the above $L_{1,\phi_2/\phi_1}(\mathbb{R}^n)$ is the Morrey space.

**Theorem 3.3** ([27] (1997)). *Let $1 < p_2 < p_1 < \infty$ and $p_1 + p_2 \le p_1 p_2$. Assume that $\phi_i$ $(i = 1, 2)$ satisfy* (2.1), (2.2), (2.3) *and* (3.1) *with $p = p_i$. Assume also that $\phi_1$ and $\phi_2$ satisfy* (3.3) *for $p = p_2$ and that $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$. If $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$ is almost increasing, then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n)) = \mathrm{BMO}_{\phi_3}(\mathbb{R}^n) \cap L_{1,\phi_2/\phi_1}(\mathbb{R}^n)$$

*and*

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathrm{BMO}_{\phi_3}(\mathbb{R}^n)} + \|g\|_{L_{1,\phi_2/\phi_1}(\mathbb{R}^n)},$$

*where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n))$.*

**Proposition 3.4** ([27] (1997)). *Suppose that $\phi_1$ and $\phi_2$ satisfy the doubling condition* (2.1). *Let $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$. If $1 \le p_2 < p_1 < \infty$ and $p_4 \ge p_1 p_2/(p_1 - p_2)$, then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n)) \supset \mathcal{L}^{\natural}_{p_2,\phi_3}(\mathbb{R}^n) \cap L_{p_4,\phi_2/\phi_1}(\mathbb{R}^n)$$

*and*

$$\|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathcal{L}_{p_2,\phi_3}} + \|g\|_{L_{p_4,\phi_2/\phi_1}}),$$

*where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n))$.*

**Corollary 3.5** ([38] (to appear)). *Let $1 \le p_2 < p_1 < \infty$ and $1/p_4 = 1/p_2 - 1/p_1$. Suppose that $\phi$ satisfies the doubling condition* (2.1) *and that there exists a positive constant $C_\phi$ such that*

$$\int_0^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi \quad \text{for all } x \in \mathbb{R}^n, \tag{3.4}$$

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi\,\phi(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \ge 1. \tag{3.5}$$

*Let*

$$\psi(x,r) = \begin{cases} \phi(x,r) & r < 1, \\ \phi(x,r)^2 & r \ge 1. \end{cases}$$

*If $f \in \mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n)$, $g \in \mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \sigma(g) = 0$, then $fg \in \mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n)$, $\sigma(fg) = 0$ and*

$$\|fg\|_{\mathcal{L}^{\natural}_{p_2,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p_1,\phi}} \|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}. \tag{3.6}$$

For example, we can take $p_1 = p_4 = 4$ and $p_2 = 2$.

## § 4. Examples of pointwise multipliers on Campanato spaces

Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman-Weiss [8, 9], i.e., $X$ is a topological space endowed with a Borel measure $\mu$ and a quasi-distance $d$ such that $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$,

$$d(x, y) \leq K_1 \, (d(x, z) + d(z, y)), \quad x, y, z \in X, \tag{4.1}$$

the balls $B(x, r) = \{y \in X; d(x, y) < r\}$ centered at $x$ and of radius $r > 0$ form a basis of open neighborhoods of the point $x$, and

$$0 < \mu(B(x, 2r)) \leq K_2 \, \mu(B(x, r)) < \infty, \quad x \in X, \; r > 0. \tag{4.2}$$

We assume that there are constants $\alpha_0$ $(0 < \alpha_0 \leq 1)$ and $K_3 \geq 1$ such that

$$|d(x, z) - d(y, z)| \leq K_3 \, (d(x, z) + d(y, z))^{1-\alpha_0} d(x, y)^{\alpha_0}, \quad x, y, z \in X. \tag{4.3}$$

If $d$ is a distance, then (4.1) and (4.3) hold for $K_1 = K_3 = \alpha_0 = 1$.

In the results in the previous section we can replace $\mathbb{R}^n$ with spaces of homogeneous type $(X, d, \mu)$. If $\mu(X) < \infty$, then we can omit the condition (3.3). For example, $X$ is a cube $Q \subset \mathbb{R}^n$ or $\mathbb{T}^n$.

### § 4.1. The case $\mu(X) < \infty$.

**Example 4.1** ([27] (1997)). For $0 \leq \beta < \alpha < 1$,

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{(\log(1/r))^{-\alpha}}(X), \mathrm{BMO}^\natural_{(\log(1/r))^{-\beta}}(X)) = \mathrm{BMO}^\natural_{(\log(1/r))^{\alpha-\beta-1}}(X).$$

For $\alpha = 1/2$ and $\beta = 0$ in particular,

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{(\log(1/r))^{-1/2}}(X), \mathrm{BMO}^\natural(X)) = \mathrm{BMO}^\natural_{(\log(1/r))^{-1/2}}(X).$$

**Example 4.2** ([27] (1997)).

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{(\log(1/r))^{-1}}(X), \mathrm{BMO}^\natural(X)) = \mathrm{BMO}^\natural_{(\log\log(1/r))^{-1}}(X).$$

**Example 4.3** ([27] (1997)).

$$\begin{aligned}
&\mathrm{PWM}(\mathrm{BMO}^\natural_{(\log\log(1/r))^{-1}}(X), \mathrm{BMO}^\natural(X)) \\
&\qquad\qquad = \mathrm{BMO}_{(\mathrm{li}(\log(1/r)))^{-1}}(X) \cap L_{1,(\log\log(1/r))}(X),
\end{aligned}$$

where $\mathrm{li}(R) = \int_e^R 1/(\log t)\, dt$.

**Example 4.4** ([27] (1997)).

$$\mathrm{PWM}(\mathrm{BMO}^\natural(X)) = \mathrm{BMO}_{(\log(1/r))^{-1}}(X) \cap L^\infty(X).$$

If $X = \mathbb{T}^n$, $d(x,y) = |x-y|$ and $\mu$ is Lebesgue measure, then the example above is known (Janson [15] and Stegenga [46]).

**Example 4.5** ([27] (1997)). For $\alpha > 1$,

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{(\log(1/r))^{-\alpha}}(X), \mathrm{BMO}^\natural(X)) = \mathrm{BMO}^\natural(X).$$

**Example 4.6** ([27] (1997)). For $0 < \beta \le \alpha \le \alpha_0$,

$$\mathrm{PWM}(\mathrm{Lip}^\natural_\alpha(X), \mathrm{Lip}^\natural_\beta(X)) = \mathrm{Lip}^\natural_\beta(X).$$

**Example 4.7** ([27] (1997)). For $-1 < \alpha < \beta \le \alpha+1,\ 1 < p_2 < p_1 < \infty,\ p_1p_2 \ge p_1 + p_2$,

$$\mathrm{PWM}(\mathcal{L}^\natural_{p_1,(\log(1/r))^\alpha}(X), \mathcal{L}^\natural_{p_2,(\log(1/r))^\beta}(X)) = \mathrm{BMO}^\natural_{(\log(1/r))^{\beta-\alpha-1}}(X).$$

## § 4.2. The case $\mu(X) = \infty$, fix $x_0 \in X$.

If $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and $\mu$ is Lebesgue measure, then we can take $x_0$ is the origin in $\mathbb{R}^n$.

**Example 4.8** ([27] (1997)).

$$\mathrm{PWM}(\mathrm{BMO}^\natural(X), \mathrm{BMO}^\natural(X)) = \mathrm{BMO}_{(\log(d(x_0,x)+r+1/r))^{-1}}(X) \cap L^\infty(X).$$

If $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and $\mu$ is Lebesgue measure, then the example above is Theorem 1.3.

**Example 4.9** ([27] (1997)). For $0 < \beta \le \alpha \le \alpha_0$,

$$\mathrm{PWM}(\mathrm{Lip}^\natural_\alpha(X), \mathrm{Lip}^\natural_\beta(X)) = \mathrm{BMO}_{\frac{r^\beta}{(2+d(x_0,x)+r)^\alpha}}(X) \cap L_{1,r^{\beta-\alpha}}(X).$$

**Example 4.10** ([27] (1997)). For $0 < \alpha \le \alpha_0,\ \beta \ge 0,\ \beta - \alpha + \delta > 0$,

$$\begin{aligned}\mathrm{PWM}(&\mathrm{BMO}^\natural_{(2+d(x_0,x)+r)^\alpha}(X), \mathrm{BMO}^\natural_{(2+d(x_0,x)+r)^\beta}(X)) \\ &= \mathrm{BMO}_{\frac{(2+d(x_0,x)+r)^{\beta-\alpha}}{\log(d(x_0,x)+r+1/r)}}(X) \cap L_{1,(2+d(x_0,x)+r)^{\beta-\alpha}}(X).\end{aligned}$$

**Example 4.11** ([27] (1997)). For $1 < p < \infty$,

$$\mathrm{PWM}(\mathrm{BMO}^\natural(X), \mathcal{L}^\natural_{p,\log(d(x_0,x)+r+1/r)}(X)) = \mathrm{BMO}^\natural(X).$$

**Example 4.12** ([27] (1997)). Let $w$ be an $A_{p'}$-weight on $\mathbb{R}^n$. Then

$$\phi(a,r) = \left(\int_{B(a,r)} w(x)\,dx\right)^{\alpha}$$

satisfies the assumption in Theorem 3.1 for $-1/(pp') < \alpha \le 1/(np')$, and is almost increasing for $\alpha \ge 0$. Let

$$\phi_i(a,r) = \left(\int_{B(a,r)} w(x)\,dx\right)^{\alpha_i}, i = 1, 2, \qquad 0 < \alpha_2 \le \alpha_1.$$

Then $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$.

## § 5. Singular integral operators

Let $0 < \kappa \le 1$. We shall consider a singular integral operator $T$ with kernel $K$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ satisfying the following properties:

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{5.1}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \frac{C}{|x-y|^n}\left(\frac{|x-z|}{|x-y|}\right)^{\kappa} \quad \text{for} \quad |x-y| \ge 2|x-z|, \tag{5.2}$$

$$\int_{r\le|x-y|<R} K(x,y)\,dy = \int_{r\le|x-y|<R} K(y,x)\,dy = 0 \quad \text{for } 0 < r < R < \infty \text{ and } x \in \mathbb{R}^n, \tag{5.3}$$

where $C$ is a positive constant independent of $x, y, z \in \mathbb{R}^n$.

For $\eta > 0$, let

$$T_\eta f(x) = \int_{|x-y|\ge\eta} K(x,y)f(y)\,dy.$$

Then $T_\eta f(x)$ is well defined for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. We assume that, for all $1 < p < \infty$, there exists positive constant $C_p$ independently $\eta > 0$ such that,

$$\|T_\eta f\|_{L^p} \le C_p\|f\|_{L^p} \quad \text{for} \quad f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n),$$

and $T_\eta f$ converges to $Tf$ in $L^p(\mathbb{R}^n)$ as $\eta \to 0$. By this assumption, the operator $T$ can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. We shall say the operator $T$ satisfying the above conditions is a singular integral operator of type $\kappa$.

Now, to define $T$ for functions $f$ in Campanato spaces we first define the modified version of $T_\eta$ by

$$\tilde{T}_\eta f(x) = \int_{|x-y|\ge\eta} f(y)\big[K(x,y) - K(0,y)(1 - \chi_{B(0,1)}(y))\big]\,dy. \tag{5.4}$$

If $\phi$ satisfies (2.1) and $\int_1^\infty \frac{\phi(x,t)}{t^2}\,dt < \infty$, then we can show that the integral in the definition above converges absolutely for each $x$ and that $\tilde{T}_\eta f$ converges in $L^p(B)$ as $\eta \to 0$ for each ball $B$. We denote the limit by $\tilde{T}f$. If both $\tilde{T}f$ and $Tf$ are well defined, then the difference is a constant.

**Theorem 5.1** ([31] (2010)). *Let $0<\kappa\le 1$ and $1<p<\infty$. Assume that $\phi$ satisfies* (2.1) *and that there exists a positive constant $A$ such that, for all $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,*

$$r^\kappa \int_r^\infty \frac{\phi(x,t)}{t^{1+\kappa}}\,dt \le A\phi(x,r). \tag{5.5}$$

*If $T$ is a singular integral operator of type $\kappa$, then $\tilde{T}$ is bounded on $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$, that is, there exists a positive constants $C$ such that*

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\phi}} \le C\|f\|_{\mathcal{L}_{p,\phi}}, \quad \|\tilde{T}f\|_{\mathcal{L}^\natural_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

*Moreover, if $\phi$ satisfies* (2.2) *and* (2.3) *also, then $\tilde{T}$ is bounded on $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{1,\phi}(\mathbb{R}^n)$.*

For example, $\phi(x,r)=r^{\lambda(x)}$ with $-n/p \le \inf_{x\in\mathbb{R}^n}\lambda(x) \le \sup_{x\in\mathbb{R}^n}\lambda(x) < 1$ satisfies the condition (5.5).

**Corollary 5.2** ([31] (2010)). *Under the assumption in Theorem 5.1, if $\phi$ and $\psi$ satisfy* (2.2), (2.3) *and* (2.5), *then $\tilde{T}$ is bounded from $\Lambda_\phi(\mathbb{R}^n)$ to $\Lambda_\psi(\mathbb{R}^n)$ and from $\Lambda^\natural_\phi(\mathbb{R}^n)$ to $\Lambda^\natural_\psi(\mathbb{R}^n)$.*

For Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$, we have the following.

**Theorem 5.3** ([26] (1994)). *Let $1<p<\infty$. Assume that $\phi$ satisfies* (2.1) *and that there exists a positive constant $A$ such that, for all $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le A\phi(x,r).$$

*If $T$ is a singular integral operator with kernel satisfying* (5.1), *and if $T$ is bounded on $L^p(\mathbb{R}^n)$, then $T$ can be extended to a bounded operator on $L_{p,\phi}(\mathbb{R}^n)$.*

For example, $\phi(x,r)=r^{\lambda(x)}$ with $-n/p \le \inf_{x\in\mathbb{R}^n}\lambda(x) \le \sup_{x\in\mathbb{R}^n}\lambda(x) < 0$ satisfies the above condition.

Next we state the boundedness of the Riesz transforms particularly, which are singular integral operators of type 1. For $f\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ the Riesz transforms of $f$ are defined by

$$R_jf(x) = \lim_{\varepsilon\to 0} R_{j,\varepsilon}f(x), \quad j=1,\dots,n,$$

where

$$R_{j,\varepsilon}f(x) = c_n \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}.$$

Then it is known that, for all $1 < p < \infty$, there exists a positive constant $C_p$ independently $\varepsilon > 0$ such that,

$$\|R_{j,\varepsilon}f\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{for} \quad f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n),$$

and $R_{j,\varepsilon}f$ converges to $R_j f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. That is, the operator $R_j$ can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. Hence, we can define a modified Riesz transforms of $f$ as

$$\tilde{R}_j f(x) = \lim_{\varepsilon \to 0} \tilde{R}_{j.\varepsilon} f(x), \quad j = 1, \dots, n,$$

and

$$\tilde{R}_{j.\varepsilon} f(x) = c_n \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \left( \frac{x_j - y_j}{|x-y|^{n+1}} - \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} \right) f(y)\, dy.$$

We note that, if both $R_j f$ and $\tilde{R}_j f$ are well defined on $\mathbb{R}^n$, then $R_j f - \tilde{R}_j f$ is a constant function. More precisely,

$$R_j f(x) - \tilde{R}_j f(x) = c_n \int_{\mathbb{R}^n} \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} f(y)\, dy.$$

**Theorem 5.4** ([38] (to appear)). *Let $1 < p < \infty$. Assume that $\phi$ satisfies* (2.1) *and that there exists a positive constant $A$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,*

$$r \int_r^\infty \frac{\phi(x,t)}{t^2}\, dt \le A\phi(x,r). \tag{5.6}$$

*Assume also that there exists a growth function $\tilde{\phi}$ such that $\phi \le \tilde{\phi}$ and that $\tilde{\phi}$ satisfies* (2.1), (2.2) *and* (2.6). *If $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then, for each $j = 1, 2, \dots, n$, $R_j f$ is well defined, $\sigma(R_j f) = \lim_{r\to\infty} (R_j f)_{B(0,r)} = 0$, and $\|R_j f\|_{\mathcal{L}^\natural_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}$, where the constant $C$ is independent of $f$.*

*Remark.* From Theorem 5.4 we conclude that, under the assumption, if $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then $R_i R_j f$ is well defined and

$$\|R_i R_j f\|_{\mathcal{L}^\natural_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}, \quad i, j = 1, \dots, n.$$

## § 6. Convolution with the heat kernel

Let

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \quad \text{for} \quad x \in \mathbb{R}^n,\ t \in (0,\infty). \tag{6.1}$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}. \tag{6.2}$$

**Theorem 6.1** ([38] (to appear)). *Let $1 \le p_2 \le p_1 < \infty$. Assume that $\phi$ satisfies (2.1) and (5.6). Then there exists a positive constant $C$ such that, for all $t \in (0,\infty)$ and $f \in \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$,*

$$\|h_t * f\|_{\mathcal{L}_{p_1,\theta}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}_{p_1,\theta}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$

*where $\theta(x,r) = (1 + r^{(1/p_2-1/p_1)n})\phi(x,r)$. Moreover, if there exists a positive constant $C_\phi$ such that, for all $x \in \mathbb{R}^n$, $\int_1^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi$, then there exists a positive constant $C$ such that, for all $t \in (0,\infty)$ and $f \in \mathcal{L}^\natural_{p_2,\phi}(\mathbb{R}^n)$,*

$$\|h_t * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}.$$

*Further, if $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \int_r^\infty \frac{\phi(x,t)}{t}\,dt = 0$, then $\sigma(f) = 0$ implies $\sigma(h_t*f) = \sigma((\nabla h_t)*f) = 0$.*

**Theorem 6.2** ([38] (to appear)). *Let $1 \le p_2 \le p_1 < \infty$. Assume that $\psi$ satisfies (2.1) and (5.6) and that there exists a positive constant $C_\psi$ such that, for all $x \in \mathbb{R}^n$, $\int_0^\infty \frac{\psi(x,t)}{t}\,dt \le C_\psi$. Assume also that $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \int_r^\infty \frac{\psi(x,t)}{t}\,dt = 0$. Let*

$$\phi(x,r) = \begin{cases} \psi(x,r) & r < 1, \\ \psi(x,r)^{p_2/p_1} & r \ge 1. \end{cases}$$

*Then, for $f \in \mathcal{L}^\natural_{p_2,\psi}(\mathbb{R}^n)$ with $\sigma(f) = 0$, then $\sigma(h_t * f) = \sigma((\nabla h_t) * f) = 0$ and*

$$\|h_t * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}.$$

## § 7. An application: The Cauchy problem for the Navier-Stokes equation

The Navier-Stokes equation is expressed as

$$
\tag{7.1}
\begin{cases}
\partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = 0 & \text{in } \mathbb{R}^n \times [0,T),\\
\nabla\cdot v = 0 & \text{in } \mathbb{R}^n \times [0,T),\\
v|_{t=0} = v_0 & \text{in } \mathbb{R}^n,
\end{cases}
$$

where $v=(v_1,\dots,v_n)$ is a vector field representing velocity of the fluid, $p$ is the pressure, and

$$
\nabla\cdot v = \sum_{j=1}^n \partial_j v_j, \quad v\cdot\nabla = \sum_{j=1}^n v_j\partial_j, \quad \Delta = \sum_{j=1}^n \partial_j^2.
$$

It is known that the pair of solutions $(v,p)$ satisfies the relation

$$
p = \sum_{i,j=1}^n R_iR_j(v_iv_j),
$$

where the operators $R_j$ $(j=1,\dots,n)$ are the Riesz transforms (see [13, 18, 37] for example). Therefore, to estimate the solutions in some function space we need the properties of the Riesz transforms and pointwise multipliers (pointwise product operators) on the function space. Namely, we need the following norm boundedness:

$$
\tag{7.2}
\|fg\|_{\mathcal{L}^{\natural}_{q,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p,\phi}},
$$

$$
\tag{7.3}
\|R_jf\|_{\mathcal{L}^{\natural}_{q,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{q,\psi}},
$$

for Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}$ and $\mathcal{L}^{\natural}_{q,\psi}$ with variable growth condition.

To solve (7.1) we consider the following equations:

$$
\begin{aligned}
u(t) &= e^{t\Delta}u_0 + Gu(t),\\
Gu(t) &= -\int_0^t \nabla e^{-(t-s)\Delta}P(u\otimes u)(s)\,ds,
\end{aligned}
$$

where $P$ is the Helmholtz projection; $P=(\delta_{jk}+R_jR_k)_{1\le j,k\le n}$. Then we also need the estimate of the convolution with the heat kernel.

Using Theorems 5.4, 6.1, 6.2 and Corollary 3.5, we have the following theorem:

**Theorem 7.1** ([38] (to appear)). *Let* $\max(2,n) < p < \infty$, $\phi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ *and*

$$
\psi(x,r) = \begin{cases}
\phi(x,r) & r<1,\\
\phi(x,r)^2 & r\ge 1.
\end{cases}
$$

*Assume that $\phi$ and $\psi$ satisfy* (2.1) *and* (5.6) *and that*

$$\int_0^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi \quad \text{for all } x \in \mathbb{R}^n,$$
$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi\,\phi(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \ge 1,$$
$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \phi(x,r) = 0.$$

*Assume also that there exists a growth function $\tilde{\psi}$ such that $\psi \le \tilde{\psi}$, that $\tilde{\psi}$ satisfies* (2.1), (2.2) *and* (2.6). *Then, for all $u_0 \in (\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n$ such that $\nabla \cdot u_0 = 0$ and $\sigma(u_0) = \lim_{r\to\infty}(u_0)_{B(0,r)} = 0$, there exist a positive constant $T$ (depending only on the norm of initial data) and a unique solution $u \in C([0,T);(\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n)$ to* (7.1).

For example, let $p > \max(2,n)$, $\alpha(\cdot) : \mathbb{R}^n \to (0,1)$, $\beta(\cdot) : \mathbb{R}^n \to [-n/p, 0)$, and let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r \le 1, \\ r^{\beta(x)}, & r > 1, \end{cases} \qquad \psi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r \le 1, \\ r^{2\beta(x)}, & r > 1, \end{cases}$$

and $\tilde{\psi}(x,r) = r^{2\beta_+}$, where $\alpha(\cdot)$, $\beta(\cdot)$ and $\beta_+$ satisfy

$$0 < \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < 1,$$
$$-n/p \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) = \beta_+ < 0.$$

Then $\phi$, $\psi$ and $\tilde{\psi}$ satisfy the assumption in Theorem 7.1.

For other applications of generalized Campanato spaces, see [37, 38].

## § 8. Generalized fractional integral operators on generalized Morrey spaces

In this section we state the boundedness of generalized fractional integral operators on generalized Morrey spaces. We also state on generalized fractional maximal operators.

In this and the next sections, we use the symbols $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n)$ instead of $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$:

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

$$\|f\|_{\mathcal{L}^{(p,\varphi)}} = \sup_B \left( \frac{1}{\varphi(B)} \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \fint_B |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{L^{(p,\varphi)}} = \sup_B \left( \frac{1}{\varphi(B)} \fint_B |f(y)|^p \, dy \right)^{1/p}.$$

Note that $\mathcal{L}_{1,\varphi}(\mathbb{R}^n) = \mathcal{L}^{(1,\varphi)}(\mathbb{R}^n)$ and $L_{p,\varphi}(\mathbb{R}^n) = L^{(p,\varphi^p)}(\mathbb{R}^n)$.

We say that $\theta$ is almost increasing (resp. almost decreasing) if there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\theta(x,r) \le C\theta(x,s) \quad (\text{resp. } C\theta(x,r) \ge \theta(x,s)), \quad \text{if } r < s.$$

In this and the next sections we consider the following classes of $\varphi$:

**Definition 8.1.** Let $\mathcal{G}^{\mathrm{dec}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that $\varphi$ is almost decreasing and that $r \mapsto \varphi(x,r)r^n$ is almost increasing. That is, there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$C\varphi(x,r) \ge \varphi(x,s), \quad \varphi(x,r)r^n \le C\varphi(x,s)s^n, \quad \text{if } r < s.$$

**Definition 8.2.** Let $\mathcal{G}^{\mathrm{inc}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that $\varphi$ is almost increasing and that $r \mapsto \varphi(x,r)/r$ is almost decreasing. That is, there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\varphi(x,r) \le C\varphi(x,s), \quad C\varphi(x,r)/r \ge \varphi(x,s)/s, \quad \text{if } r < s.$$

If $\varphi \in \mathcal{G}^{\mathrm{dec}}$ or $\varphi \in \mathcal{G}^{\mathrm{inc}}$, then $\varphi$ satisfies the doubling condition (2.1).

First we state the boundedness of the Hardy-Littlewood maximal operator $M$. It is defined by the following: For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B$ containing $x$.

**Theorem 8.3** ([32] (2014)). *Let $1 < p < \infty$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Then the operator $M$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself.*

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we consider generalized fractional integral operators $I_\rho$ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\,dy, \tag{8.1}$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t}\,dt < \infty \quad \text{for each } x \in \mathbb{R}^n, \tag{8.2}$$

and that there exist positive constants $C$, $K_1$ and $K_2$ with $K_1 < K_2$ such that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\sup_{r \le t \le 2r} \rho(x,t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t}\,dt. \tag{8.3}$$

The assumption (8.2) is needed so that $I_\rho f$ is well defined for all $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. The condition (8.3) comes from [41, p. 664 (D)].

If $\rho(x,r) = r^\alpha$, then $I_\rho$ is the usual fractional integral operator $I_\alpha$. It is known as the Hardy-Littlewood-Sobolev theorem that $I_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p+\alpha = -n/q$. If $\alpha(\cdot) : \mathbb{R}^n \to (0,n)$ and $\rho(x,r) = r^{\alpha(x)}$, then $I_\rho$ is a generalized fractional integral operator $I_{\alpha(x)}$ with variable order defined by

$$I_{\alpha(x)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy.$$

**Theorem 8.4** ([32] (2014)). *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\rho$ satisfies* (8.2) *and* (8.3) *and that $\varphi$ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies*

$$\lim_{r\to 0} \varphi(x,r) = \infty, \quad \lim_{r\to\infty} \varphi(x,r) = 0. \tag{8.4}$$

*Assume also that there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C\varphi(x,r)^{1/q}. \tag{8.5}$$

*Then $I_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

To prove the theorem above we use Hedberg's method in [14] and the boundedness of the Hardy-Littlewood maximal operator.

Next, we consider fractional maximal operators. For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let

$$M_\rho f(x) = \sup_{B \ni x} \rho(B) ⨍_B |f(y)|\,dy, \quad x \in \mathbb{R}^n, \tag{8.6}$$

where the supremum is taken over all balls $B$ containing $x$. We do not postulate the condition (8.2) or (8.3) on the definition of $M_\rho$. The operator $M_\rho$ was studied by Sawano, Sugano and Tanaka [43] (2011) on Morrey spaces in case of $\rho : (0,\infty) \to (0,\infty)$. If $\rho(B) = |B|^{\alpha/n}$, then $M_\rho$ is the usual fractional maximal operator $M_\alpha$. If $\rho \equiv 1$, then $M_\rho$ is the Hardy-Littlewood maximal operator $M$.

If $\rho(x,r)/r^n \le C\rho(x,s)/s^n$ for $0 < s < r < \infty$, then

$$M_\rho f(x) \le C I_\rho |f|(x), \quad x \in \mathbb{R}^n. \tag{8.7}$$

Hence, the boundedness of $M_\rho$ follows from the boundedness of $I_\rho$. For example, the Hardy-Littlewood-Sobolev theorem yields that $M_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. However, for example, if

$$\rho(x,r) = \begin{cases} (\log(e+1/r))^{-\beta} & (0 < r < 1) \\ (\log(e+r)^{\gamma} & (r \ge 1), \end{cases} \qquad \beta > 1,\ \gamma > 0,$$

then it turns out that the boundedness of $M_\rho$ is better than the boundedness of $I_\rho$ by the following theorem. Actually, (8.5) cannot be replaced by (8.8), see [12, Theorem 1.1].

**Theorem 8.5** ([1] (2018)). *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi$ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies* (8.4)*. Assume also that there exists a positive constant $C_0$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0 \varphi(x,r)^{1/q}. \tag{8.8}$$

*Then $M_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

## § 9. Commutators of integral operators with functions in Campanato spaces with variable growth condition

It is known that any Calderón-Zygmund operator $T$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. In 1976 Coifman, Rochberg and Weiss [7] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where $C$ is a positive constant independent of $b$ and $f$. For the fractional integral operator $I_\alpha$, Chanillo [6] proved the boundedness of $[b, I_\alpha]$ in 1982. That is,

$$\|[b,I_\alpha]f\|_{L^q} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. These results were extended to Morrey spaces by Di Fazio and Ragusa [10] in 1991.

In this section we state the boundedness of the commutators $[b, T]$ and $[b, I_\rho]$ on generalized Morrey spaces with variable growth condition, where $T$ is a Calderón-Zygmund operator, $I_\rho$ is a generalized fractional integral operator and $b$ is a function in generalized Campanato spaces with variable growth condition.

First we recall the definition of Calderón-Zygmund operators following [47]. Let $\Omega$ be the set of all nonnegative nondecreasing functions $\omega$ on $(0,\infty)$ such that $\int_0^1 \frac{\omega(t)}{t}dt < \infty$.

**Definition 9.1** (standard kernel of type $\omega$). Let $\omega \in \Omega$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type $\omega$ if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{9.1}$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\!\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| \le |x-y|. \tag{9.2}$$

**Definition 9.2** (Calderón-Zygmund operator). Let $\omega \in \Omega$. A linear operator $T$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type $\omega$, if $T$ is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel $K$ of type $\omega$ such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \quad x \notin \operatorname{supp} f. \tag{9.3}$$

It is known by [47, Theorem 2.4] that any Calderón-Zygmund operator of type $\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

This result was extended to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function $\varphi$ by [26] as the following: Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)}{t}\,dt \le C\varphi(x,r). \tag{9.4}$$

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define $Tf$ on each ball $B$ by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} K(x,y)f(y)\,dy, \quad x \in B. \tag{9.5}$$

Then the first term in the right hand side is well defined, since $f\chi_{2B} \in L^p(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball containing $x$. By this definition we can show that $T$ is a bounded operator on $L^{(p,\varphi)}(\mathbb{R}^n)$. For the definition of $Tf$, see also [35, Section 5] and [42].

For functions $f$ in Morrey spaces, we define $[b,T]f$ on each ball $B$ by

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B. \tag{9.6}$$

Then we have the following theorem.

**Theorem 9.3** ([1] (2018))**.** *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Let $T$ be a Calderón-Zygmund operator of type $\omega \in \Omega$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$.*

(i) *Assume that $\psi$ satisfies* (2.2)*, that $\varphi$ satisfies* (9.4)*, that $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$ and that there exists a positive constant $C_0$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\psi(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{9.7}$$

*If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b,T]f$ in* (9.6) *is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant $C$, independent of $b$ and $f$, such that*

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that $\varphi$ satisfies* (2.2) *and that there exists a positive constant $C_0$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$C_0\psi(x,r)\varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{9.8}$$

*If $T$ is a convolution type such that*

$$Tf(x) = p.v.\int_{\mathbb{R}^n} K(x-y)f(y)\,dy \tag{9.9}$$

*with homogeneous kernel $K$ satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, and if $[b,T]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and there exists a positive constant $C$, independent of $b$, such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

*where $\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ is the operator norm of $[b,T]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

In the above theorem, if $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p = q$. This is Coifman, Rochberg and Weiss's result in [7]. If $\psi(x,r) = r^\alpha$, $0 < \alpha \le 1$, and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ with $-n/p + \alpha = -n/q$. That is,

$$\|[b,T]f\|_{L^q} \lesssim \|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^p}.$$

This is Janson's result in [16, Lemma 12].

**Example 9.4** ([1] (2018)). Let $1 < p \le q < \infty$ and $\beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$0 \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1,$$
$$-n \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0.$$

Let

$$\psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \le r < \infty. \end{cases}$$

Let $T$ be a Calderón-Zygmund operator of type $\omega \in \Omega$ with $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$. If $\beta(\cdot)$ is log-Hölder continuous and

$$\beta(x) + \lambda(x)/p \ge \lambda(x)/q, \quad \beta_* + \lambda_*/p \le \lambda_*/q,$$

then $\psi$ and $\phi$ satisfy the assumption in Theorem 9.3 (i) and then the inequality

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}$$

holds. Conversely, if $\lambda(\cdot)$ is log-Hölder continuous and

$$\beta(x) + \lambda(x)/p \le \lambda(x)/q, \quad \beta_* + \lambda_*/p \ge \lambda_*/q,$$

and if $T$ is a convolution type with homogeneous kernel $K$ satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, then we have

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}.$$

We also consider the cases

$$\psi(x,r) = \begin{cases} r^{\beta(x)}(1/\log(e/r))^{\beta_1(x)}, & 0 < r < 1, \\ r^{\beta_*}(\log(er))^{\beta_{**}}, & 1 \le r < \infty, \end{cases}$$

etc.

For the commutator $[b, I_\rho]$ we have the following theorem.

**Theorem 9.5** ([1] (2018)). *Let $1 < p < q < \infty$ and $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that $\rho$ satisfies* (8.2) *and* (8.3). *Let $\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\,dt$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

(i) *Assume that* $\rho$, $\rho^*$ *and* $\psi$ *satisfy* (2.2), *that* $\varphi$ *satisfies* (9.4) *and that there exist positive constants* $\epsilon$, $C_\rho$, $C_0$, $C_1$ *and an exponent* $\tilde{p} \in (p,q]$ *such that, for all* $x, y \in \mathbb{R}^n$ *and* $r, s \in (0,\infty)$,

$$C_\rho \frac{\rho(x,r)}{r^{n-\epsilon}} \ge \frac{\rho(x,s)}{s^{n-\epsilon}}, \quad \text{if } r < s, \tag{9.10}$$

$$\left| \frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n} \right| \le C_\rho \left(|r-s| + |x-y|\right) \frac{\rho^*(x,r)}{r^{n+1}}, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2 \text{ and } |x-y| < r/2, \tag{9.11}$$

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C_0 \varphi(x,r)^{1/\tilde{p}}, \tag{9.12}$$

$$\psi(x,r)\varphi(x,r)^{1/\tilde{p}} \le C_1 \varphi(x,r)^{1/q}. \tag{9.13}$$

*If* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, *then* $[b, I_\rho]f$ *is well defined for all* $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* $C$, *independent of* $b$ *and* $f$, *such that*

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* $\varphi$ *satisfies* (2.2), *that* $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, *and that*

$$C_0 \psi(x,r) r^\alpha \varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \tag{9.14}$$

*If* $[b, I_\alpha]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$, *then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* $C$, *independent of* $b$, *such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C \|[b, I_\alpha]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}},$$

*where* $\|[b, I_\alpha]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}$ *is the operator norm of* $[b, I_\alpha]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $I_\rho = I_\alpha$, $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. This is Chanillo's result in [6]. See also [20].

**Example 9.6** ([1] (2018))**.** Let $1 < p < \tilde{p} \le q < \infty$ and $\alpha(\cdot), \beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$0 < \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < n, \quad 0 < \alpha_* < n,$$

$$0 \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1,$$

$$-n \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0.$$

Let

$$\rho(x,r)=\begin{cases} r^{\alpha(x)}, \\ r^{\alpha_*}, \end{cases} \quad \psi(x,r)=\begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r)=\begin{cases} r^{\lambda(x)}, & 0<r<1, \\ r^{\lambda_*}, & 1\le r<\infty. \end{cases}$$

If $\alpha(\cdot)$ is Lipschitz continuous, $\beta(\cdot)$ is log-Hölder continuous and

$$\begin{aligned} &\sup_{x\in\mathbb{R}^n}(\alpha(x)+\lambda(x)/p)<0, \\ &\alpha(x)+\lambda(x)/p\ge\lambda(x)/\tilde{p}, \quad \alpha_*+\lambda_*/p\le\lambda_*/\tilde{p}, \\ &\beta(x)+\lambda(x)/\tilde{p}\ge\lambda(x)/q, \quad \beta_*+\lambda_*/\tilde{p}\le\lambda_*/q, \end{aligned}$$

then

$$\|[b,I_\rho]f\|_{L^{(q,\varphi)}}\le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Conversely, if $\lambda(\cdot)$ is log-Hölder continuous, $\alpha$ is constant and

$$\alpha+\beta(x)+\lambda(x)/p\le\lambda(x)/q, \quad \alpha+\beta_*+\lambda_*/p\ge\lambda_*/q,$$

then

$$\|b\|_{\mathcal{L}^{(1,\psi)}}\le C\|[b,I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}.$$

We also take the cases

$$\rho(x,r)=\begin{cases} r^{\alpha(x)}(1/\log(e/r))^{\alpha_1(x)}, & 0<r<1, \\ r^{\alpha_*}(\log(er))^{\alpha_{**}}, & 1\le r<\infty, \end{cases}$$

$$\rho(x,r)=\begin{cases} r^{\alpha(x)}, & 0<r<1, \\ e^{-(r-1)}, & 1\le r<\infty, \end{cases}$$

etc.

To prove Theorems above we use the following three propositions and a corollary.

For $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$M^\sharp f(x)=\sup_{B\ni x}⨍_B|f(y)-f_B|\,dy, \quad x\in\mathbb{R}^n, \tag{9.15}$$

where the supremum is taken over all balls $B$ containing $x$.

**Proposition 9.7** ([1] (2018)). *Let $p,\eta\in(1,\infty)$, $\varphi\in\mathcal{G}^{\mathrm{dec}}$ and $\psi\in\mathcal{G}^{\mathrm{inc}}$. Let $T$ be a Calderón-Zygmund operator of type $\omega$. Assume that $\psi$ satisfies* (2.2)*, that $\varphi$ satisfies* (9.4)*, that $\int_0^1\frac{\omega(t)\log(1/t)}{t}\,dt<\infty$ and that $\int_r^\infty\frac{\psi(x,t)\varphi(x,t)^{1/p}}{t}\,dt<\infty$ for each $x\in\mathbb{R}^n$ and $r>0$. Then there exists a positive constant $C$ such that, for all $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f\in L^{(p,\varphi)}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,*

$$M^\sharp[b,T]f(x)\le C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(M_{\psi^\eta}(|Tf|^\eta)(x)\big)^{1/\eta}+\big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big). \tag{9.16}$$

**Proposition 9.8** ([1] (2018)). *Let $p, \eta \in (1, \infty)$, $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume that $\rho : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ satisfies* (8.2) *and* (8.3). *Let $\rho^*(x, r) = \int_0^r \frac{\rho(x,t)}{t}\, dt$. Assume that $\rho$, $\rho^*$ and $\psi$ satisfy* (2.2), *that $\varphi$ satisfies* (9.4) *and that there exist positive constants $\epsilon$, $C_\rho$ such that* (9.10) *and* (9.11) *hold. Assume also that*

(9.17)
$$\int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\, dt < \infty, \quad \int_r^\infty \frac{\psi(x,t)}{t} \left( \int_t^\infty \frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\, du \right) dt < \infty,$$

*for each $x \in \mathbb{R}^n$ and $r > 0$. Then there exists a positive constant $C$ such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^\sharp([b, I_\rho]f)(x) \le C\|b\|_{\mathcal{L}^{(1,\psi)}} \left( \big(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\big)^{1/\eta} + \big(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\big)^{1/\eta} \right). \tag{9.18}$$

**Proposition 9.9** ([1] (2018)). *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. If $\varphi$ satisfies the doubling condition* (2.1), *then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{L}^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{9.19}$$

*where $C$ is a positive constant independent of $f$.*

By Theorem 2.5 we have the following corollary.

**Corollary 9.10** ([1] (2018)). *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that $\varphi$ satisfies* (9.4). *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\lim_{r\to\infty} f_{B(0,r)} = 0$, then*

$$\|f\|_{L^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{9.20}$$

*where $C$ is a positive constant independent of $f$.*

Then, using Propositions 9.7 and 9.8, Corollary 9.10 and the boundedness of $T$, $I_\rho$ and $M_\rho$ (Theorems 5.3, 8.4 and 8.5, respectively), we have

$$\|[b, T]f\|_{L^{(q,\varphi)}} \lesssim \|M^\sharp([b, T]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}},$$
$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \lesssim \|M^\sharp([b, I_\rho]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

These shows Theorem 9.3 (i) and Theorem 9.5 (i). The parts (ii) in Theorems 9.3 and 9.5 are proved by Janson's method in [16].

We also have the compactness of $[b, T]$ and $[b, I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$, see [2, 3].

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